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LETTER TO THE EDITOR

Heisenberg *XXZ* model and quantum Galilei group

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Abstract. The 1D Heisenberg spin model with anisotropy of the *XXZ* type is analysed in terms of the symmetry given by the quantum Galilei group $\Gamma_q(1)$. For a chain with an infinite number of sites we show that the magnon excitations and the $s = 1/2$, n-magnon bound states are determined by the algebra. In this case the $\Gamma_q(1)$ symmetry provides a description naturally compatible with the Bethe ansatz. The recurrence relations determined by $\Gamma_q(1)$ permit us to express the energy of the n-magnon bound states in a closed form in terms of Tchebischeff polynomials.

Very recently some promising results have been obtained by the application of the symmetry of inhomogeneous quantum groups [1] to physical systems with a fundamental scale. In [2] the rotational spectra of heavy nuclei have been reproduced, while in [3] and [4] applications to solid state problems have been studied. In the former case the deformation parameter of the quantum group is related to the time scale of strong interactions, while in the latter a fundamental length arises naturally from the lattice spacing. In [4] we have shown that the symmetry described by the q-deformation of the Galilei group in one dimension, $\Gamma_q(1)$, yields an algebraic scheme consistent with the Bethe ansatz [5, 6] for solving the dynamics of quantum integrable models. The method has been illustrated using the concrete example of the infinite length isotropic (or *XXX*) Heisenberg ferromagnet. In this letter we shall use $\Gamma_q(1)$ for studying a magnetic chain with anisotropy of *XXZ* type, whose properties have been throughly investigated [7, 8]. We shall also show that the conditions that are to be imposed on the mass of the composite systems for obtaining bound states are connected with the critical behaviour of the Casimir operator of $\Gamma_q(1)$.

The Hamiltonian of the model with periodic conditions $S_{N+1} = S_1$, ($S = S^x, S^y, S^z$), is given by [7]

$$\mathcal{H} = 2J \sum_{i=1}^{N} \left((1-\alpha)\left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y\right) + S_i^z S_{i+1}^z \right). \tag{1}$$

Let $|0\rangle$ be the state with all the spin directed downwards. This is an eigenstate of $\mathcal{H}$ with an energy given by $\epsilon_0 = 2JNs^2$. In terms of the states with one spin deviate, $\psi = \sum_i f_i S_i^+ |0\rangle$, the eigenvalue equation for $\mathcal{H}$ translates into the algebraic system

$$2Js\left((1-\alpha)(f_{i-1} + f_{i+1}) - 2f_i\right) = (\epsilon - \epsilon_0) f_i \tag{2}$$

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which leads to the dispersion relation

$$\epsilon - \epsilon_0 = -4Js\big(1 - (1-\alpha)\cos k\big)\,. \tag{3}$$

We have shown in [4] that the isotropic analogue of system (2) can be described by means of a quantum group symmetry. Indeed, the solutions of (2) are obtained by evaluating the solutions of the differential equation

$$-4Js\big(1-(1-\alpha)\cos(-\mathrm{i}a\partial_x)\big)f(x) = (\epsilon-\epsilon_0)f(x) \tag{4}$$

at integer multiples of the lattice spacing a. This form of the Schrödinger equation on the lattice has been proposed, e.g., in [9]. For $a \to 0$, we recover from (4) the stationary Schrödinger equation with an effective mass $(-4Js\,(1-\alpha)\,a^2)^{-1}$ and with the symmetry of the 1D Galilei group. We were therefore led in [4] to introduce the deformation $\Gamma_q(1)$ of the Galilei algebra, generated by the four elements B, M, P, T with commutation relations

$$[B,P] = \mathrm{i}M \qquad [B,T] = (\mathrm{i}/a)\sin(aP) \qquad [P,T] = 0$$

the generator M being central.

The coproducts and the antipodes read

$$\Delta B = \mathrm{e}^{-\mathrm{i}aP}\otimes B + B\otimes \mathrm{e}^{\mathrm{i}aP} \qquad \Delta M = \mathrm{e}^{-\mathrm{i}aP}\otimes M + M\otimes \mathrm{e}^{\mathrm{i}aP}$$

$$\Delta P = \mathbf{1}\otimes P + P\otimes \mathbf{1} \qquad \Delta T = \mathbf{1}\otimes T + T\otimes \mathbf{1}$$

$$\gamma(B) = -B - aM \qquad \gamma(M) = -M \qquad \gamma(P) = -P \qquad \gamma(T) = -T$$

while the Casimir of $\Gamma_q(1)$ is

$$C = MT - (1/a^2)\,(1-\cos(aP))\,. \tag{5}$$

This quantum algebra admits the following realization in terms of differential operators

$$B = mx \qquad M = m \qquad P = -\mathrm{i}\partial_x$$

$$T = (ma^2)^{-1}\big(1-\cos(-\mathrm{i}a\partial_x)\big) + c/m$$

where c is the constant value of the Casimir: for $(ma^2)^{-1} = -4Js\,(1-\alpha)$ and $c/m = -4Js\alpha$, the expression of T coincides with the operator on the left hand side of (4).

Like the isotropic model [4], the algebra is invariant under $P \mapsto P + (2\pi/a)\,n$, the position operator is defined as $X = (1/M)\,B$ and the properties of the one-magnon states are obtained from the $\Gamma_q(1)$ symmetry.

The properties of the two-magnon states $\psi = \sum_{i>j} f_{ij}\; S_i^+ S_j^+ |0\rangle$, with $f_{ij} = f_{ji}$, are then described by the following system in the coefficients f_{ij}

$$\big(\epsilon - \epsilon_0 + 8Js\big) f_{ij} - 2s(1-\alpha)\sum_n \big(J_{nj}\,f_{in} + J_{in}\,f_{nj}\big)$$

$$= -J_{ij}\big((1-\alpha)\,(f_{ii}+f_{jj}) - f_{ij} - f_{ji}\big)\,. \tag{6}$$

The bonds J_{ij} are equal to J when the label (ij) represents nearest neighbour pairs and vanish otherwise. For $s = 1/2$ the amplitudes f_{ii} cancel in pairs and $\Gamma_q(1)$, which is a symmetry of the free system, permits a complete treatment of two-magnon excitations. Indeed, owing to the Bethe ansatz, which imposes the separate vanishing of the two sides of equation (6), the interaction is reduced to 'boundary conditions' ensuring that the homogeneous free equation is satisfied at every pair of sites.

In our previous papers [2–4] we have already proved that the coproduct is the correct operation which allows an algebraic treatment of the many excitation systems. From ΔT we therefore find the following two-magnon energy T_{12}

$$\begin{aligned} T_{12} &= T_1 + T_2 \\ &= (M_1 a^2)^{-1}(1 - \cos(aP_1)) + (M_2 a^2)^{-1}(1 - \cos(aP_2)) + U_1 + U_2 \end{aligned} \tag{7}$$

with $U_i = C_i/M_i$, $i = 1, 2$. For a fixed value of the spin s, the eigenvalue of $M_1 = M_2 = M$ is equal to $(-4Js(1-\alpha)a^2)^{-1}$ and $U_1 = U_2 = -4Js\alpha$. Using the differential realization $P_1 = -i\partial_{x_1}$ and $P_2 = -i\partial_{x_2}$, the action of T_{12} on the two-magnon amplitude $f(x_1, x_2)$ reads

$$\begin{aligned} T_{12} f(x_1, x_2) = -8Js\, f(x_1, x_2) + 2Js(1-\alpha)\big(&f(x_1, x_2 + a) \\ &+ f(x_1, x_2 - a) + f(x_1 + a, x_2) + f(x_1 - a, x_2)\big). \end{aligned} \tag{8}$$

The eigenvalue equation for T_{12} is equivalent to the vanishing of the left hand side of equation (6). Plane waves solve this eigenvalue equation and the energy of the continuum is

$$\epsilon - \epsilon_0 = -4Js\big(2 - (1-\alpha)\cos(ap_1) - (1-\alpha)\cos(ap_2)\big).$$

The two-magnon state eigenfunctions are then obtained by imposing both the periodicity and the Bethe boundary conditions. We also observe that the coproduct of P gives $P_{12} = P_1 + P_2$ for the total momentum which, in the *XXZ* model, is easily seen to be a conserved quantity.

Let us now show how the bound states for the infinite length chain and $s = 1/2$ can be obtained from the $\Gamma_q(1)$ symmetry. We first observe that the central generator M for the composite system reads

$$M_{12} = M_1 e^{iaP_2} + M_2 e^{-iaP_1}.$$

We then consider the energy for the two-magnon system by rewriting (7) in the form

$$T_{12} = (a^2 M_{12})^{-1}(1 - \cos(aP_{12})) + U_{12} \tag{9}$$

where

$$U_{12} = U_1 + U_2 - \frac{(M_{12} - M_1 - M_2)^2}{2a^2 M_{12} M_1 M_2}. \tag{10}$$

From equations (5), (9) and (10) the coproduct of the Casimir gives

$$C_{12} = M_{12} U_{12}.$$

The operators C_{12} and M_{12} label the irreducible representations of the composite system and therefore must assume the same value over each state of a given irreducible representation. We shall now show that the critical behaviour of C_{12}, as a function of the global mass M_{12}, defines the Bethe conditions for the bound states and thus their energy. Indeed $\partial C_{12}/\partial M_{12} = 0$ gives

$$M_{12} = M_1 + M_2 + a^2 M_1 M_2 (U_1 + U_2)\,. \tag{11}$$

In the continuum Galilei limit we find that M_{12} is identically equal to $M_1 + M_2$, while, for $\alpha = 0$ (i.e. for $U_1 = U_2 = 0$), we recover the analogous condition for the isotropic *XXX* model [4]. For two $s = 1/2$ magnons, in the above notation, equation (11) reads

$$M_{12} = 2M/(1-\alpha)\,. \tag{12}$$

Defining $2iv = P_1 - P_2$ the last condition yields just the Bethe ansatz for bound states in the limit of an infinite number of sites [7]

$$e^{-v} = (1-\alpha)\cos(aP/2)\,.$$

By substituting equation (12) into (9) and (10) we get the known form for the energy of bound states

$$T_{12} = -2J\left(1 - (1-\alpha)^2 \cos^2(aP/2)\right)\,.$$

We now give the generalization to the n-magnon case. The total energy obtained from the quantum group can be written as

$$T_{12\ldots n} = \sum_{k=1}^{n} T_k = (a^2 M_{12\ldots n})^{-1}\left(1 - \cos(aP_{12\ldots n})\right) + U_{12\ldots n} \tag{13}$$

where $P_{12\ldots n} = \sum_{k=1}^{n} P_k$ and

$$U_{12\ldots n} = \sum_{k=1}^{n} U_k - \frac{1}{2a^2} \sum_{k=2}^{n} \frac{(M_{12\ldots k} - M_{12\ldots(k-1)} - M_k)^2}{M_{12\ldots k} M_{12\ldots(k-1)} M_k}\,. \tag{14}$$

In the above equations $M_{12\ldots k}$ are defined by iterating the coproduct and using the coassociativity

$$M_{l\ldots k} = M_{l\ldots(h-1)}\, e^{ia(P_h + \cdots + P_k)} + M_{h\ldots k}\, e^{-ia(P_l + \cdots + P_{h-1})} \qquad l < h \leqslant k\,. \tag{15}$$

Moreover the coproducts of the Casimir $C_{12\ldots k}$ are found to be

$$C_{12\ldots k} = M_{12\ldots k}\, U_{12\ldots k}\,.$$

The bound states are obtained from (13) and (14) by imposing vanishing of the sequence of the derivatives of $C_{12\ldots k}$ with respect to $M_{12\ldots k}$ for $k = 2, \ldots, n$. The conditions determining $M_{12\ldots k}$ read

$$M_{12\ldots k} = M_{12\ldots(k-1)} + M_k + a^2 M_{12\ldots(k-1)} M_k (U_{12\ldots(k-1)} + U_k) \qquad k = 2, \ldots, n\,. \tag{16}$$

We have performed a computer-assisted analysis [10] of the recurrence relations (14) and (16) and we have found that they can be solved, yielding

$$M_{12\ldots k} = -\left(2J(1-\alpha)a^2\right)^{-1} U_{k-1}(1/(1-\alpha)) \qquad k = 2, \ldots, n \tag{17}$$

$$T_{12\ldots n} = \frac{-2J(1-\alpha)}{U_{n-1}(1/(1-\alpha))}\left(\mathcal{T}_n(1/(1-\alpha)) - \cos(aP_{12\ldots k})\right) \tag{18}$$

where U_k and $\mathcal{T}_k$ are the Tchebischeff polynomials [11]. The equations (17), where $M_{12\ldots k}$ have the form given in (15), are equivalent to the Bethe conditions

$$M_{(k-1)k} = 2M/(1-\alpha) = -\left(J(1-\alpha)^2 a^2\right)^{-1} \qquad k = 2\ldots, n\,.$$

Equation (18) gives the energy of the n-magnon bound states.

Some final remarks are in order. In the first place we observe that the treatment given in [4] of the *XXX* model has been here extended in a straightforward way also to the anisotropic *XXZ* model. We thus provide further evidence for the significance of the application of the inhomogeneous quantum groups, like $\Gamma_q(1)$ and $E_q(1,1)$, as kinematical symmetries of elementary physical systems described by a discretized Schrödinger or Klein–Gordon equation: it is the appropriate quantum group that indicates the Bethe ansatz and then the integrability of the system, so that explicit computations are made possible.

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